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Green's function of bimaterials comprising all cases of material degeneracy

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Abstract

For bimaterials with planar interfaces subjected to a line force and dislocation, Green's functions are determined for all types of anisotropic materials including the nondegenerate, degenerate and extra-degenerate cases. The changes in Green's function caused by material degeneracy are twofold: (i) implicit changes, attributable to material effects only and characterized by high-order eigenvectors and their intrinsic coupling in the higher-order eigensolutions; (ii) explicit changes, influenced by boundary and interface conditions, that cause additional terms in Green's function. Material degeneracy affects the angular variation of the singular stress field, which may have significant implication on the failure prediction of strongly anisotropic materials. For *all* material types, Green's functions are obtained for bimaterials with a planar interface, and for multi-material wedges subjected to a line force and dislocation at the vertex. The results are expressed in a concise notation in terms of the complete set of eigenvectors and kernel matrices of analytic functions.

Keywords: Green's function; Bimaterials; Anisotropic elasticity; Degenerate materials; Singularities and dislocations

1. Introduction

Degenerate and extra-degenerate anisotropic materials pose special problems in elasticity analysis, because the usual representation of two-dimensional general solutions in terms of three complex conjugate pairs of material eigenvectors ceases to be valid. Such materials have a conjugate pair of *multiple* eigenvalues, which may possess a smaller number of independent eigenvectors than the multiplicity of the eigenvalue, so that higher-order eigensolutions must be found to make up for the deficiency. General solutions

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of all types of anisotropic materials, including degenerate and extra-degenerate cases, have been given explicitly in recent works in both compliance-based and stiffness-based formalisms (Yin, 2000a,b). Solutions of elasticity problems, including Green's functions for the various domains, require the representations of general solutions appropriate to the specific type of material. This accounts for the paucity of previous elasticity solutions for degenerate and extra-degenerate materials, except in two special cases: (i) when the material is isotropic and (ii) when the domain of the problem is a half space, an infinite space with or without a crack, or two dissimilar half spaces either perfectly bonded or separated by a semi-infinite crack. In the first case, the Goursat representation in terms of two complex analytic functions does give the general solution of plane isotropic elasticity (excluding antiplane deformation, which is uncoupled from in-plane solutions). However, both the general solution and the particular solutions of specific problems in isotropic elasticity are analytically more complicated than the corresponding problems of a nondegenerate anisotropic material, due to the intrinsic coupling of the zeroth- and higher-order eigenvectors. In the second case involving half-space domains, the effects of material degeneracy are partially circumvented, because some important results of the elasticity solution may be formulated in terms of the Stroh–Barnett–Lothe tensors in a manner insensitive to degeneracy (Ting, 1996). For domains of general shapes, the key features of the solution, including the interfacial stresses and displacements, and the power of the singularity at the vertex of a multi-material wedge, cannot be characterized by the Stroh–Barnett–Lothe tensors alone. Then the form and complexity of the elasticity solution depend essentially on the material types.

Two-dimensional general solutions of degenerate and extra-degenerate materials (to be abbreviated as D and ED materials, respectively) were used in a recent paper to determine Green's functions of a number of domains subjected to a line force or dislocation (Yin, 2004a). The domains examined include an infinite space, a half space, and the exterior region of an elliptical cylinder. For the half-space problem, a more general type of homogeneous boundary condition was considered than the usual assumptions of traction-free boundaries or fixed boundaries. The resulting expressions of Green's functions are more complex than the well-known results for nondegenerate (ND) materials. Since all isotropic materials belong to one family of degenerate materials, the solutions for this family naturally yield Green's functions of isotropic materials as a particular case. On the other hand, attempts to deduce Green's functions of isotropic materials from the general solutions of *nondegenerate* materials (for example, Choi et al., 2003) cannot succeed, since the latter do not contain the higher-order eigenvectors that are essential to the representation of the solutions of a degenerate material. Similarly, Green's functions of ED materials contain additional terms that do not appear in the expressions for degenerate materials.

The present analysis relies on general solutions of ND, D and ED materials given in recent works on anisotropic plane elasticity, and some key results are recapitulated in Sections 2 and 3. In Section 4, we obtain Green's function of bimaterials consisting of two perfectly bonded half spaces with any combination of ND, D and ED materials. The forms of expression are suggested by the previous solutions of Green's functions for the half-space of unrestricted material types (Yin, 2004a). The effect of material degeneracy on the analytic form of Green's function of a bimaterial and on the singular part of its expression are examined in detail. Explicit expressions of Green's functions of anisotropic bimaterials are shown in Appendix A, including the known results for ND materials and new solutions for D and ED cases.

By differentiating Green's functions with respect to the position of singularity, some related Green's functions with higher-order singularities are easily obtained. Additional derived results are found by a limiting process when an interior singularity is allowed to move to the boundary or the interface (Section 6). More generally, one may obtain special Green's function of a multimaterial wedge composed of dissimilar sectors joined along radial interfaces and subjected to a line load or dislocation at the vertex. The stress and strain fields in each sector are given by Green's function of the infinite space for the material of that sector. The present solutions again extend the previous results of ND materials (Ting, 1996) to the D and ED cases.

2. Eigenvectors and eigensolutions; normal and abnormal materials

Let $\alpha_{ij}(i,j=1,\ldots,6)$ denote the anisotropic elastic compliance constants relating the strain components ε_x, ε_y, ε_z, γ_{yz}, γ_{xz}, γ_{xy} to the stress components σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy}, and let (Lekhnitskii, 1963)

$$\beta_{ij} = \alpha_{ij} - \alpha_{i3}\alpha_{j3}/\alpha_{33} \quad (\text{for } i,j \neq 3), \tag{2.1}$$

Then, for generalized plane deformation (whose stress and strain are independent of z), one has

$$\{\varepsilon\} = [\beta]\{\sigma\}, \tag{2.2}$$

where $\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^{\mathrm{T}}$, $\{\sigma\} = \{\sigma_x, \sigma_y, \tau_{yz}, \tau_{xz}, \tau_{xy}\}^{\mathrm{T}}$ and $[\beta]$ is the 5×5 symmetric matrix of the reduced compliance coefficients β_{ij}. In the absence of body forces, the equilibrium conditions imply that $\{\sigma\}$ may be represented by the derivatives of a pair of stress functions $F(x,y)$ and $\Psi(x,y)$

$$\sigma_x = F_{,yy}, \quad \sigma_y = F_{,xx}, \quad \tau_{xy} = -F_{,xy}, \quad \tau_{xz} = \Psi_{,y}, \quad \tau_{yz} = -\Psi_{,x}. \tag{2.3}$$

We seek solutions for the displacements $\mathbf{u} = \{u, v, w\}$ and the stress potentials $\mathbf{q} = \{F_{,y}, -F_{,x}, \Psi\}$ of the following form:

$$\mathbf{q} = \mathbf{b}f(z,\mu), \quad \mathbf{u} = \mathbf{a}f(z,\mu), \quad z \equiv x + \mu y, \tag{2.4a, b, c}$$

where the scalar function f is analytic in the first argument z, and the complex parameter μ affects f explicitly as the second argument and implicitly through z. The strain and stress fields associated with Eqs. (2.4a,b) are easily obtained by differentiation. Using $\tau_{xy} = -\partial_x F_{,y} = -b_1 f'(x+\mu y) = \partial_y(-F_{,x}) = b_2\mu f'(x+\mu y)$, one has $b_1 = -\mu b_2$. These results in conjunction with the strain–displacement relation yield

$$\{\varepsilon\} = f'(x+\mu y)\mathbf{E}(\mu)\mathbf{a}, \quad \{\sigma\} = f'(x+\mu y)\mathbf{P}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}, \tag{2.5}$$

where the matrix functions $\mathbf{E}(\mu)$ and $\mathbf{P}(\mu)$ are defined by

$$\mathbf{E}(\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 1 \\ \mu & 1 & 0 \end{bmatrix}, \quad \mathbf{P}(\mu) = \begin{bmatrix} -\mu^2 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & \mu \\ \mu & 0 \end{bmatrix}. \tag{2.6a, b}$$

Using the identity $\mathbf{P}^{\mathrm{T}}\mathbf{E} = \mathbf{0}$ and $\mathbf{E}^{\mathrm{T}}\mathbf{P} = \mathbf{0}$, one obtains from Eqs. (2.2) and (2.5) the eigenrelation

$$\mathbf{E}(\mu)\mathbf{a} = [\beta]\mathbf{P}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}. \tag{2.7}$$

Pre-multiplication of the last equation by $\mathbf{P}(\mu)^{\mathrm{T}}$ yields,

$$\mathbf{M}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} = \mathbf{0}, \tag{2.8}$$

where the matrix function $\mathbf{M}(\mu)$ and its adjoint matrix $\mathbf{W}(\mu)$ have the expressions

$$\mathbf{M}(\mu) = \mathbf{P}^{\mathrm{T}}(\mu)[\beta]\mathbf{P}(\mu) = \begin{bmatrix} l_4(\mu) & -l_3(\mu) \\ -l_3(\mu) & l_2(\mu) \end{bmatrix}, \quad \mathbf{W}(\mu) = \begin{bmatrix} l_2(\mu) & l_3(\mu) \\ l_3(\mu) & l_4(\mu) \end{bmatrix} \tag{2.9a, b}$$

with

$$l_2(\mu) = \beta_{44} - 2\beta_{45}\mu + \beta_{55}\mu^2, \quad l_3(\mu) = -\beta_{24} + (\beta_{25}+\beta_{46})\mu - (\beta_{14}+\beta_{56})\mu^2 + \beta_{15}\mu^3,$$
$$l_4(\mu) = \beta_{22} - 2\beta_{26}\mu + (2\beta_{12}+\beta_{66})\mu^2 - 2\beta_{16}\mu^3 + \beta_{11}\mu^4.$$

Eq. (2.8) has a nontrivial solution for $\{b_2, b_3^{\mathrm{T}}\}^{\mathrm{T}}$ if and only if μ satisfies

$$\delta(\mu) = \mid \mathbf{M}(\mu) \mid = \mid \mathbf{W}(\mu) \mid = l_2(\mu)l_4(\mu) - l_3(\mu)^2 = 0. \tag{2.10}$$

Eqs. (2.9a,b) yield the identity (where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix)

$$\mathbf{M}(\mu)\mathbf{W}(\mu) = \delta(\mu)\mathbf{I}_2. \tag{2.11}$$

If μ is a root of Eq. (2.10), then Eq. (2.8) has a nontrivial solution $\{b_2, b_3\}^{\mathrm{T}}$, and it may be verified straightforwardly that the following expressions satisfy Eq. (2.7):

$$\mathbf{b} = \begin{bmatrix} -\mu & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} [\beta]\mathbf{P}(\mu) \begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}. \tag{2.12a, b}$$

The roots of the characteristic equation (2.10) are the *material eigenvalues*. The eigenvalues occur in complex conjugate pairs. The eigenvalues cannot be real if the strain energy density is positive definite. For each eigenvalue, Eqs. (2.12a,b) give a pair of **b**- and **a**-vectors which determine a zeroth-order *eigenvector* $\boldsymbol{\xi}^{[0]}$ and a zeroth-order eigensolution $\boldsymbol{\chi}^{[0]}$

$$\boldsymbol{\xi}^{[0]} = \begin{Bmatrix} \mathbf{b} \\ \mathbf{a} \end{Bmatrix} = \mathbf{J}(\mu) \begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}, \quad \boldsymbol{\chi}^{[0]} = \begin{Bmatrix} \mathbf{q} \\ \mathbf{u} \end{Bmatrix} = f(z, \mu)\boldsymbol{\xi}^{[0]}, \tag{2.13a, b}$$

where

$$\mathbf{J}(\mu) \equiv \begin{Bmatrix} \mathbf{J}_1(\mu) \\ \mathbf{J}_2(\mu) \end{Bmatrix}, \quad \mathbf{J}_1(\mu) \equiv \begin{bmatrix} -\mu & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{J}_2(\mu) \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\mu & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} [\beta]\mathbf{P}(\mu). \tag{2.14a, b, c}$$

It is easily shown that the eigenvector and eigensolution associated with the conjugate eigenvalue $\bar{\mu}$ are, respectively, the complex conjugates of $\boldsymbol{\xi}^{[0]}$ and $\boldsymbol{\chi}^{[0]}$.

Anisotropic materials are classified into five distinct types depending on the multiplicity of eigenvalues, and on whether the multiple eigenvalues are *normal* or *abnormal*. The classification is important because the general solution of plane elasticity assumes different algebraic forms for different types of materials. An eigenvalue μ is called *normal* if $\mathbf{M}(\mu)$ is *not* the null matrix. For a normal $\mu, \mathbf{W}(\mu)$ has one and only one independent column. Such a column, denoted by $\boldsymbol{\eta}(\mu)$, gives a nontrivial solution $\{b_2, b_3\}^{\mathrm{T}}$ of Eq. (2.8). With Eq. (2.14a,b,c), it yields an eigenvector $\boldsymbol{\xi}^{[0]} = \mathbf{J}(\mu)\boldsymbol{\eta}(\mu)$ and an eigensolution $\boldsymbol{\chi}^{[0]} = f(z, \mu)\mathbf{J}(\mu)\boldsymbol{\eta}(\mu)$. An eigenvalue μ is called *abnormal* if it is not normal. Then it must be a multiple eigenvalue, and Eq. (2.8) is satisfied by any pair of independent 2-vectors such as $\{1, 0\}^{\mathrm{T}}$ and $\{0, 1\}^{\mathrm{T}}$.

If the characteristic equation has a repeated root μ whose multiplicity is greater than the number of independent solutions of Eq. (2.8), then the material is called *degenerate* or *extra-degenerate*, depending on whether the deficiency in the independent eigenvectors is 1 or 2. In such cases, the set of zeroth-order eigenvectors must be supplemented by higher-order ones.

For normal and abnormal eigenvalues of multiplicity p, the complete set of p independent eigenvectors and eigensolutions are given as follows:

2.1. Eigenvectors and eigensolutions associated with a normal eigenvalue

If μ^* is a *normal* eigenvalue of multiplicity $p \geqslant 2$, there are p independent eigensolutions $\boldsymbol{\chi}^{[0]}, \ldots, \boldsymbol{\chi}^{[p-1]}$ expressed by

$$\boldsymbol{\chi}^{[k]} = \mathrm{d}^k/\mathrm{d}\mu^k\{f_{k+1}(z, \mu)\mathbf{J}(\mu)\mathbf{W}(\mu)\boldsymbol{\rho}\}|_{\mu=\mu^*} \quad (0 \leqslant k \leqslant p-1), \tag{2.15a}$$

where the column selector $\boldsymbol{\rho}$ is a column of $\mathbf{I}_2$ such that $\mathbf{W}(\mu^*)\boldsymbol{\rho}$ is not the null vector. If we set $f_k(z,\mu)=1$ in Eq. (3.13a) for $k=1,\ldots,p$, we obtain a corresponding set of eigenvectors $\boldsymbol{\xi}^{[0]},\boldsymbol{\xi}^{[1]},\ldots,\boldsymbol{\xi}^{[p-1]}$

$$\boldsymbol{\xi}^{[k]}=\mathrm{d}^k/\mathrm{d}\mu^k\{\mathbf{J}(\mu)\mathbf{W}(\mu)\}|_{\mu=\mu^*}\quad(0\leqslant k\leqslant p-1).\tag{2.15b}$$

The eigenvectors and eigensolutions are related by

$$\boldsymbol{\chi}^{[k]}=\sum_{0\leqslant j\leqslant k}\{k!/j!(k-j)!\}\{\mathrm{d}^{(k-j)}f_{k+1}/\mathrm{d}\mu^{(k-j)}\}\boldsymbol{\xi}^{[j]}.\tag{2.15c}$$

2.2. Eigenvectors and eigensolutions associated with an abnormal eigenvalue

For an *abnormal* eigenvalue μ^* of multiplicity $p\geqslant 2$, a complete set of eigensolutions is

$$\begin{aligned}&\boldsymbol{\chi}^{[0]}=f_1(x+\mu^*y,\mu^*)\mathbf{J}(\mu^*)\{0,1\}^{\mathrm{T}},\\&\boldsymbol{\chi}^{[k]}=\mathrm{d}^k/\mathrm{d}\mu^k(f_{k+1}(z,\mu)\mathbf{J}(\mu)\{l_2(\mu),l_3(\mu)\}^{\mathrm{T}})|_{\mu=\mu^*}\quad(1\leqslant k\leqslant p-1).\end{aligned}\tag{2.16a}$$

The eigenvectors

$$\begin{aligned}&\boldsymbol{\xi}^{[0]}=\mathbf{J}(\mu^*)\{0,1\}^{\mathrm{T}},\\&\boldsymbol{\xi}^{[k]}=\mathrm{d}^k/\mathrm{d}\mu^k(\mathbf{J}(\mu)\{l_2(\mu),l_3(\mu)\}^{\mathrm{T}})|_{\mu=\mu^*}\quad(1\leqslant k\leqslant p-1)\end{aligned}\tag{2.16b}$$

are related to the eigensolutions by the equations

$$\begin{aligned}&\boldsymbol{\chi}^{[0]}=f_1(x+\mu^*y,\mu^*)\boldsymbol{\xi}^{[0]},\\&\boldsymbol{\chi}^{[k]}=\sum_{1\leqslant j\leqslant k}\{k!/j!(k-j)!\}\{\mathrm{d}^{(k-j)}f_{k+1}/\mathrm{d}\mu^{(k-j)}|_{\mu=\mu^*}\}\boldsymbol{\xi}^{[j]}\quad(1\leqslant k\leqslant p-1).\end{aligned}\tag{2.16c}$$

2.3. The base matrix Z

Let $\{\mu\}_\perp$ denote the sequence of all three eigenvalues with positive imaginary parts, such that simple eigenvalues precede any multiple eigenvalue. This sequence is followed by its complex conjugate $\{\bar{\mu}\}_\perp$ to form the complete set $\{\mu\}$. All eigenvectors will be obtained strictly according to Eq. (2.15b) or (2.16b) for a normal and an abnormal eigenvalue, respectively, and arranged as the column vectors of a 6×6 *base matrix* $\mathbf{Z}$ consisting of two 6×3 submatrices $\mathbf{Z}_\perp$ and $\overline{\mathbf{Z}}_\perp$. Those eigenvectors $\boldsymbol{\xi}^{[k]}$ that belong to a common eigenvalue are arranged in the order of increasing k. One has

$$\mathbf{Z}\equiv[\mathbf{Z}_\perp,\overline{\mathbf{Z}}_\perp]=\begin{bmatrix}\mathbf{B}&\overline{\mathbf{B}}\\\mathbf{A}&\overline{\mathbf{A}}\end{bmatrix},\tag{2.17}$$

where $\mathbf{B}$ and $\mathbf{A}$ are 3×3 submatrices of $\mathbf{Z}_\perp$. In order to obtain real-valued $\boldsymbol{\chi}$, the analytic functions in Eqs. (2.15c) and (2.16c) will be required to satisfy

$$f_{k+3}(x+\bar{\mu}y)\equiv\overline{f_k(x+\mu y)}\quad(k=1,2,3).\tag{2.18}$$

3. General solutions of the five distinct types of anisotropic materials

There are five distinct types of anisotropic materials, characterized by the respective sets of eigenvalues:

ND-normal. Three simple eigenvalues μ_1, μ_2 and μ_3.
ND-abnormal. One simple eigenvalue μ_1 and one abnormal double eigenvalue μ_0.
D-normal. One simple eigenvalue μ_1 and one normal double eigenvalue μ_0.
D-abnormal. One abnormal triple eigenvalue μ_0.
ED. One normal triple eigenvalue μ_0.

Any two eigenvectors, $\xi^{[k]}$ and $\xi'^{[j]}$, associated with distinct eigenvalues μ and μ', are shown to be orthogonal in the following sense regardless of whether the eigenvectors are of the zeroth- or higher-orders:

$$(\xi^{[k]})^{\mathrm{T}}\mathbf{II}\xi'^{[j]} = 0, \tag{3.1}$$

where

$$\mathbf{II} \equiv \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0}_{3\times3} \end{bmatrix} \tag{3.2}$$

and the subscripts attached to the identity and zero matrices indicate their dimensions. Eq. (3.1) implies that

$$\overline{\mathbf{Z}}_\perp^{\mathrm{T}}\mathbf{II}\mathbf{Z}_\perp = \mathbf{Z}_\perp^{\mathrm{T}}\mathbf{II}\overline{\mathbf{Z}}_\perp = \mathbf{A}^{\mathrm{T}}\overline{\mathbf{B}} + \mathbf{B}^{\mathrm{T}}\overline{\mathbf{A}} = \mathbf{0}. \tag{3.3}$$

We define symmetric matrices $\boldsymbol{\Omega}_\perp$ and $\boldsymbol{\Omega}$ of dimensions 3×3 and 6×6, respectively

$$\boldsymbol{\Omega}_\perp \equiv \mathbf{Z}_\perp^{\mathrm{T}}\mathbf{II}\mathbf{Z}_\perp, \tag{3.4}$$

$$\boldsymbol{\Omega} \equiv \mathbf{Z}^{\mathrm{T}}\mathbf{II}\mathbf{Z} = \langle \boldsymbol{\Omega}_\perp, \overline{\boldsymbol{\Omega}}_\perp \rangle. \tag{3.5}$$

Here the symbol $\langle\,\rangle$ denotes a block-diagonal matrix containing a number of diagonal blocks separated by commas. For an ND material, all eigenvectors are zeroth-order and mutually orthogonal, so that $\boldsymbol{\Omega}$ is a diagonal matrix. For D and ED materials, $\boldsymbol{\Omega}$ is block diagonal, where each diagonal block is a nonsingular submatrx associated with a distinct eigenvalue (Yin, 2000a,b). Then $\boldsymbol{\Omega}$ is also nonsingular and so is $\mathbf{Z}$, in view of Eq. (3.5). Hence, regardless of material degeneracy, the six eigenvectors in $\mathbf{Z}$ are independent. One has

$$\boldsymbol{\Omega}^{-1} = \langle \boldsymbol{\Omega}_\perp^{-1}, \overline{\boldsymbol{\Omega}}_\perp^{-1} \rangle, \quad \mathbf{Z}^{-1} = \boldsymbol{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{II}. \tag{3.6a, b}$$

Explicit analytical expressions of $\boldsymbol{\Omega}_\perp$ and $\boldsymbol{\Omega}_\perp^{-1}$ may be found in Yin (2000a) for all five classes of ND-, D- and ED-materials. Now let

$$\boldsymbol{\Gamma} \equiv \begin{bmatrix} -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \\ \mathbf{S} & \mathbf{H} \end{bmatrix} \equiv \mathbf{Z}\langle -\mathrm{i}\mathbf{I}_3, \mathrm{i}\mathbf{I}_3 \rangle \mathbf{Z}^{-1}\mathbf{II} = \mathbf{Z}\langle -\mathrm{i}\mathbf{I}_3, \mathrm{i}\mathbf{I}_3 \rangle \boldsymbol{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}}. \tag{3.7}$$

Clearly, $\boldsymbol{\Gamma}$ is a symmetric matrix. Since $\mathbf{Z} = [\mathbf{Z}_\perp, \overline{\mathbf{Z}}_\perp]$, the last expression of (3.7) is equal to its complex conjugate. Hence $\boldsymbol{\Gamma}$ and the 3×3 submatrices $\mathbf{H}, \mathbf{L}$ and $\mathbf{S}$ (the Stroh–Barnett–Lothe tensors) are all real and $\mathbf{L}$ and $\mathbf{H}$ are symmetric. Eq. (3.7) also yields

$$-\boldsymbol{\Gamma}\mathbf{II}\boldsymbol{\Gamma}\mathbf{II} = \begin{bmatrix} \mathbf{LH} - (\mathbf{S}^{\mathrm{T}})^2 & \mathbf{LS} + \mathbf{S}^{\mathrm{T}}\mathbf{L} \\ -\mathbf{SH} - \mathbf{HS}^{\mathrm{T}} & \mathbf{HL} - \mathbf{S}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix}. \tag{3.8}$$

Hence $\mathbf{HL} - \mathbf{S}^2 = \mathbf{LH} - (\mathbf{S}^{\mathrm{T}})^2 = \mathbf{I}_3$ and $\mathbf{SH}$ and $\mathbf{LS}$ are skew-symmetric. Then so must be $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{SL}^{-1}$. Eq. (3.6a,b) may be rewritten as $\mathbf{Z}\boldsymbol{\Omega}^{-1}\mathbf{Z}^{\mathrm{T}} = \mathbf{II}$, or $\mathrm{Re}[\mathbf{Z}_\perp\boldsymbol{\Omega}_\perp^{-1}\mathbf{Z}_\perp^{\mathrm{T}}] = (1/2)\mathbf{II}$, whereas Eq. (3.7) yields $\mathrm{Im}[\mathbf{Z}_\perp\boldsymbol{\Omega}_\perp^{-1}\mathbf{Z}_\perp^{\mathrm{T}}] = (1/2)\,\mathbf{G}$. Whence follows the useful result:

$$\mathbf{Z}_{\perp}\boldsymbol{\Omega}_{\perp}^{-1}\mathbf{Z}_{\perp}^{\mathrm{T}} = (1/2)(\mathbf{II} + \mathrm{i}\boldsymbol{\Gamma}) = (1/2)\begin{bmatrix} -\mathrm{i}\mathbf{L} & \mathbf{I}_3 + \mathrm{i}\mathbf{S}^{\mathrm{T}} \\ \mathbf{I}_3 + \mathrm{i}\mathbf{S} & \mathrm{i}\mathbf{H} \end{bmatrix}. \tag{3.9}$$

The 2-D general solution may be written in the following concise form for all five types of ND, D and ED materials:

$$\boldsymbol{\chi} = \mathbf{Z}\|f(x+\mu y, \mu)\|\mathbf{c}, \tag{3.10}$$

where $\mathbf{c}$ is a six-dimensional complex constant vector such that

$$c_{j+3} = \bar{c}_j \quad (j = 1, 2, 3) \tag{3.11}$$

and

$$\|f(x+\mu y, \mu)\| = \langle \|f(x+\mu y, \mu)\|_{\perp}, \# \rangle \tag{3.12}$$

is the kernel matrix of analytic functions. Here and elsewhere, $\langle \boldsymbol{\sigma}, \# \rangle$ indicates the block diagonal matrix containing a square matrix $\boldsymbol{\sigma}$ and its complex conjugate matrix as the diagonal blocks: $\langle \boldsymbol{\sigma}, \# \rangle \equiv \langle \boldsymbol{\sigma}, \overline{\boldsymbol{\sigma}} \rangle$. The first 3×3 diagonal block of the kernel matrix is given by the following expressions for the ND, D-normal, D-abnormal and ED cases, respectively

$$\|f(x+\mu y, \mu)\|_{\perp} = \langle f_1(z_1, \mu_1), f_2(z_2, \mu_2), f_3(z_3, \mu_3) \rangle, \tag{3.13a}$$

$$\|f(z, \mu)\|_{\perp} = \begin{bmatrix} f_1(z_1, \mu_1) & 0 & 0 \\ 0 & f_2(z_0, \mu_0) & \{yf_{3,z} + f_{3,\mu}\}(z_0, \mu_0) \\ 0 & 0 & f_3(z_0, \mu_0) \end{bmatrix}, \tag{3.13b}$$

$$\|f(z, \mu)\|_{\perp} = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 2(yf_{3,z} + f_{3,\mu}) \\ 0 & 0 & f_3 \end{bmatrix}, \tag{3.13c}$$

$$\|f(z, \mu)\|_{\perp} = \begin{bmatrix} f_1 & yf_{2,z} + f_{2,\mu} & y^2 f_{3,zz} + 2yf_{3,z\mu} + f_{3,\mu\mu} \\ 0 & f_2 & 2(yf_{3,z} + f_{3,\mu}) \\ 0 & 0 & f_3 \end{bmatrix}, \tag{3.13d}$$

where $z \equiv x + \mu y$ and $z_k \equiv x + \mu_k y$ $(k = 1, 2, 3)$. In Eqs. (3.13c,d), all three functions f_1,f_2,f_3 and their various derivatives are evaluated at $z = z_0 \equiv x + \mu_0 y$ and at the triple eigenvalue μ_0.

4. Two-dimensional Green's functions of bimaterials with a planar interface

Consider two different anisotropic materials, occupying the upper and lower half spaces, that are perfectly bonded along the interface $y = 0$. The materials of the upper and lower regions have the eigenvalues μ_j and μ'_j, respectively $(j = 1, 2, 3)$, along with their complex conjugates, and $\mathrm{Im}[\mu_j]$ and $\mathrm{Im}[\mu'_j]$ are positive. Unless both materials are nondegenerate, the base matrices

$$\mathbf{Z} \equiv \begin{bmatrix} \mathbf{B} & \overline{\mathbf{B}} \\ \mathbf{A} & \overline{\mathbf{A}} \end{bmatrix}, \quad \mathbf{Z}' \equiv \begin{bmatrix} \mathbf{B}' & \overline{\mathbf{B}}' \\ \mathbf{A}' & \overline{\mathbf{A}}' \end{bmatrix}, \tag{4.1}$$

generally contain higher-order eigenvectors. A singularity exists at $(x,y)=(b,h)$ of the upper medium, such that $\boldsymbol{\chi}=\{F_{,y},-F_{,x},\Psi,u,v,w\}$ increases by a constant amount $2\pi\boldsymbol{\chi}_0$ when one makes a complete circle around the point (b,h) in the clockwise sense, regardless of the radius of the circle. The first three and the last three components of $\boldsymbol{\chi}_0$ correspond, respectively, to the three components of a concentrated line force and of a line dislocation. The stress field vanishes at infinity. Let $\boldsymbol{\chi}'$ denote the stress potentials and the displacements in the lower region. Continuity of the tractions across $y=0$ implies that the stress potentials $F_{,y}$, $-F_{,x}$ and Ψ of the two media may differ at most by constant values on the interface. Green's function of the bimaterial is defined to be the pair of 6×6 matrix functions $\mathbf{G}(x,y)$ and $\mathbf{G}'(x,y)$ which transform the constant vector $\boldsymbol{\chi}_0$ into the vector fields $\boldsymbol{\chi}$ and $\boldsymbol{\chi}'$, respectively

$$\boldsymbol{\chi}=\mathbf{G}\boldsymbol{\chi}_0,\quad \boldsymbol{\chi}'=\mathbf{G}'\boldsymbol{\chi}_0. \tag{4.2}$$

On $y=0$, the boundary conditions of the half space is characterized by a 3×6 *real* matrix $\mathbf{K}$ with integer elements 0 or 1

$$\mathbf{KG}|_{y=0}=\mathbf{0}. \tag{4.3}$$

A 3×3 complex matrix $\mathbf{T}$ is defined in terms of $\mathbf{K}$ and $\mathbf{Z}$

$$\mathbf{T}\equiv(\mathbf{KZ}_\perp)^{-1}\mathbf{K}\overline{\mathbf{Z}}_\perp. \tag{4.4}$$

Let

$$z=x+\mu y,\quad z'=x+\mu' y,\quad z_0\equiv x+\mu_0 y, \tag{4.5}$$

$$\rho=b+\mu h,\quad \rho_0\equiv b+\mu_0 h,\quad \rho_j\equiv b+\mu_j h,\bar\rho_j\equiv b+\bar\mu_j h\quad (j=1,2,3) \tag{4.6}$$

where μ_0 denotes the multiple eigenvalue of a D or ED material. Furthermore, for $i\neq j$, let Δ_{ij} denote the 3×3 matrices with all elements zero except the element 1 in the ith row and the jth column. We will also use Δ_j to denote the 3×3 diagonal matrix with all elements zero except the jth diagonal element which is 1.

The various expressions of Green's function of the half space with the boundary condition (4.3) are given as follows (Yin, 2004a):

Case (i)
ND materials

$$\begin{aligned}\mathbf{G}&=\mathbf{G}_{\mathrm{ND}}\\&\equiv\sum_{1\leqslant j\leqslant3}\mathbf{Z}\langle-\mathrm{i}\log[z_j-\rho_j]\mathbf{I}_3,\#\rangle\langle\Delta_j,\Delta_j\rangle\mathbf{Z}^{-1}+\sum_{1\leqslant j\leqslant3}\mathbf{Z}\|-\mathrm{i}\log[z_j-\bar\rho_j]\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\langle\Delta_j,\Delta_j\rangle\mathbf{IIZ}^{-1},\end{aligned} \tag{4.7a}$$

Case (ii)
D-normal material

$$\begin{aligned}\mathbf{G}=\;&\mathbf{G}_{\mathrm{ND}}+\mathbf{Z}\langle-\mathrm{i}(y-h)(z_0-\rho_0)^{-1}\mathbf{I}_3,\#\rangle\langle\Delta_{23},\Delta_{23}\rangle\mathbf{Z}^{-1}\\&+\mathbf{Z}\|\mathrm{i}h(z-\bar\rho_0)^{-1}\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\langle\Delta_{23},\Delta_{23}\rangle\mathbf{IIZ}^{-1},\end{aligned} \tag{4.7b}$$

Case (iii)
D-abnormal material. $\mathbf{G}$ is given by Eq. (4.7b) with Δ_{23} replaced by $2\Delta_{23}$, i.e.,

$$\begin{aligned}\mathbf{G}=\;&\mathbf{Z}\|-\mathrm{i}\log[z-\rho]\|\mathbf{Z}^{-1}+\mathbf{Z}\|-\mathrm{i}\log[z-\bar\rho_0]\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\mathbf{IIZ}^{-1}\\&+\mathbf{Z}\|\mathrm{i}h(z-\bar\rho_0)^{-1}\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\langle2\Delta_{23},2\Delta_{23}\rangle\mathbf{IIZ}^{-1},\end{aligned} \tag{4.7c}$$

Case (iv)
ED material

$$\mathbf{G} = \mathbf{G}_{\mathrm{ND}} + \mathbf{Z}\langle -\mathrm{i}(y-h)(z_0-\rho_0)^{-1}\mathbf{I}_3, \#\rangle\langle \varDelta_{12}+2\varDelta_{23}, \varDelta_{12}+2\varDelta_{23}\rangle\mathbf{Z}^{-1} + \mathbf{Z}\langle \mathrm{i}(y-h)^2(z_0-\rho_0)^{-2}\mathbf{I}_3, \#\rangle\langle \varDelta_{13}, \varDelta_{13}\rangle\mathbf{Z}^{-1} + \mathbf{Z}\|\mathrm{i}h(z-\bar{\rho})_0^{-1}\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle\langle \varDelta_{12}+2\varDelta_{23}, \varDelta_{12}+2\varDelta_{23}\rangle\mathbf{IIZ}^{-1} + \mathbf{Z}\|\mathrm{i}h^2(z-\bar{\rho}_0)^{-2}\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle\langle \varDelta_{13}, \varDelta_{13}\rangle\mathbf{IIZ}^{-1}. \tag{4.7d}$$

The kernel matrices in these expressions are as defined by Eqs. (3.13a–d) for the various cases. All terms on the right-hand sides of (4.7a–d) that end with postmultiplication by $\mathbf{Z}^{-1}$ only belongs to $\mathbf{G}_\infty$, which is Green's function of an infinite space:

$$\mathbf{G}_\infty = \mathbf{Z}\| -\mathrm{i}\log[z-\rho]\|\mathbf{Z}^{-1}. \tag{4.8}$$

In this primary group, z_j occurs only with ρ_j and $\bar{z}_j$ with $\bar{\rho}_j$. The remaining terms end with post-multiplication by $\mathbf{IIZ}^{-1}$, and in these terms z_j occurs with $\bar{\rho}_j$ and $\bar{z}_j$ with ρ_j. They form a secondary group. To each term in the primary group corresponds a term of the secondary group which may be obtained by the following rules of replacement:

$$\begin{aligned} &\mathbf{Z}\langle -\mathrm{i}\log[z_j-\rho_j]\mathbf{I}_3, \#\rangle \to \mathbf{Z}\| -\mathrm{i}\log[z-\bar{\rho}_j]\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle, \\ &\mathbf{Z}\langle -\mathrm{i}(y-h)(z_0-\rho_0)^{-1}\mathbf{I}_3, \#\rangle \to \mathbf{Z}\|\mathrm{i}h(z-\bar{\rho}_0)^{-1}\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle, \\ &\mathbf{Z}\langle \mathrm{i}(y-h)^2(z_0-\rho_0)^{-2}\mathbf{I}_3, \#\rangle \to \mathbf{Z}\|\mathrm{i}h^2(z-\bar{\rho}_0)^{-2}\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle, \end{aligned} \tag{4.9}$$

$$\mathbf{Z}^{-1} \to \mathbf{IIZ}^{-1}. \tag{4.10}$$

As the material changes from type ND to D or ED, additional terms occur in the fully explicit expression of Eq. (4.8) due to the off-diagonal elements in $\|-\mathrm{i}\log[z-\rho]\|$. To each additional term, the rules of Eq. (4.9) yield a corresponding term to be added to Green's function of the half space to ensure the satisfaction of the boundary conditions $\mathbf{KG}|_{y=0} = \mathbf{0}$. The resulting expressions of $\mathbf{G}$ may be written in a single expression applicable to all types of materials

$$\mathbf{G} = \mathbf{Z}\| -\mathrm{i}\log[z-\rho]\|\mathbf{Z}^{-1} + \mathbf{Z}[\![\| -\mathrm{i}\log[z-\bar{\rho}]\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle]\!]\mathbf{IIZ}^{-1}, \tag{4.11}$$

where the double bracket symbol $[\![\,]\!]$ indicates (i) performing essentially the same operation on the enclosed object (as a function of $\bar{\rho}$) as the kernel matrix symbol $\|\,\|$ does to its enclosed object (as a function of z), and (ii) followed by the substitution of h for $h-y$, as in Eq. (4.9).

For the bimaterial problem, Eq. (4.3) is replaced by the interface condition

$$(\mathbf{G}-\mathbf{G}')|_{y=0} = \mathbf{0}. \tag{4.12}$$

It will be shown that, in all four cases (i)–(iv), Green's function of the bimaterial has formally the same expression as Eqs. (4.7a–d) for the elastic half space, but $\mathbf{T}$ is not given by Eq. (4.4) and, instead, depends on the materials of both regions. Furthermore, the expressions of $\mathbf{G}'$ in the various cases are formally analogous to those of $\mathbf{G}-\mathbf{G}_\infty$, except for the substitution of $z'_j, \rho_j, \mathbf{T}'$ and $\mathbf{Z}^{-1}$ for z_j, $\bar{\rho}_j$, $\mathbf{T}$ and $\mathbf{IIZ}^{-1}$, respectively, where $\mathbf{T}'$ is another bimaterial matrix to be determined. Thus, for the cases (i)–(iv), respectively, one obtains the expressions of $\mathbf{G}'$ as given by Eqs. (A.1b), (A.2b), (A.3b) and (A.4b) in Appendix A.

On $y=0$, all kernel matrices except $\|-\mathrm{i}\log[z-\rho]\|$ in the first term of $\mathbf{G}$ reduce to diagonal matrices, and Eqs. (4.7d) and (A.4b) become

$$\begin{aligned}\mathbf{G}|_{y=0} = {} & \mathbf{Z}\langle -\mathrm{i}\log[x-\rho_0]\mathbf{I}_3, \#\rangle\mathbf{Z}^{-1} + \mathbf{Z}\langle \mathrm{i}h(x-\rho_0)^{-1}\mathbf{I}_3, \#\rangle\langle \varDelta_{12}+2\varDelta_{23}, \varDelta_{12}+2\varDelta_{23}\rangle\mathbf{Z}^{-1} \\ & + \mathbf{Z}\langle \mathrm{i}h^2(x-\rho_0)^{-2}\mathbf{I}_3, \#\rangle\langle \varDelta_{13}, \varDelta_{13}\rangle\mathbf{Z}^{-1} + \mathbf{Z}\langle \mathrm{i}\log[x-\bar{\rho}_0]\mathbf{T}, \#\rangle\mathbf{II}\mathbf{Z}^{-1} \\ & + \mathbf{Z}\langle \mathrm{i}h(x-\bar{\rho}_0)^{-1}\mathbf{T}, \#\rangle\langle \varDelta_{12}+2\varDelta_{23}, \varDelta_{12}+2\varDelta_{23}\rangle\mathbf{II}\mathbf{Z}^{-1} \\ & + \mathbf{Z}\langle \mathrm{i}h^2(x-\bar{\rho}_0)^{-2}\mathbf{T}, \#\rangle\langle \varDelta_{13}, \varDelta_{13}\rangle\mathbf{II}\mathbf{Z}^{-1}. \end{aligned} \tag{4.13a}$$

$$\begin{aligned}\mathbf{G}'|_{y=0} = {} & \mathbf{Z}'\langle -\mathrm{i}\log[x-\rho_0]\mathbf{T}', \#\rangle\mathbf{Z}^{-1} + \mathbf{Z}'\langle \mathrm{i}h(x-\rho_0)^{-1}\mathbf{T}', \#\rangle\langle \varDelta_{12}+2\varDelta_{23}, \varDelta_{12}+2\varDelta_{23}\rangle\mathbf{Z}^{-1} \\ & + \mathbf{Z}'\langle \mathrm{i}h^2(x-\rho_0)^{-2}\mathbf{T}', \#\rangle\langle \varDelta_{13}, \varDelta_{13}\rangle\mathbf{Z}^{-1}. \end{aligned} \tag{4.13b}$$

Since all terms have real values, the group of terms in Eq. (4.13a) ending with $\mathbf{IIZ}^{-1}$ may be replaced by their complex conjugate expressions without affecting the result. When (4.13b) is subsequently subtracted from (4.13a), the six groups of terms containing the factors $\log[x-\rho_0]$, $(x-\rho_0)^{-1}$, $(x-\rho_0)^{-2}$ and their complex conjugate functions must vanish separately. This requires that (after using $\overline{\mathbf{Z}}^{-1} = \mathbf{IIZ}^{-1}$ which follows from $\overline{\mathbf{Z}} = \mathbf{ZII}$)

$$\mathbf{Z}_\perp - \overline{\mathbf{Z}}_\perp\overline{\mathbf{T}} = \mathbf{Z}'_\perp\mathbf{T}'. \tag{4.14}$$

The equation has the solution for $\mathbf{T}'$ and $\mathbf{T}$:

$$\begin{bmatrix}\mathbf{T}' \\ \overline{\mathbf{T}}\end{bmatrix} = \mathbf{Z}_{12}^{-1}\mathbf{Z}_\perp, \tag{4.15}$$

provided that the following *mixed* base matrix is nonsingular:

$$\mathbf{Z}_{12} \equiv [\mathbf{Z}'_\perp, \overline{\mathbf{Z}}_\perp]. \tag{4.16}$$

To prove this,we let

$$\mathbf{Z}_{21} \equiv [\mathbf{Z}_\perp, \overline{\mathbf{Z}}'_\perp] = \overline{\mathbf{Z}}_{\mathbf{12}}\mathbf{II} \tag{4.17}$$

and define two related matrices $\underline{\mathbf{Z}}_{12}$ and $\underline{\mathbf{Z}}_{21}$ in terms of $\mathbf{B}^{-1\mathrm{T}}$, $\mathbf{A}^{-1\mathrm{T}}$, $(\mathbf{B}'^{-1})^{\mathrm{T}}$ and $(\mathbf{A}'^{-1})^{\mathrm{T}}$ in exactly the same way that $\mathbf{Z}_{12}$ and $\mathbf{Z}_{21}$ have been defined in terms of $\mathbf{B}, \mathbf{A}, \mathbf{B}'$ and $\mathbf{A}'$. Then

$$\mathbf{Z}_{21}\langle \mathbf{I}_3, \mathbf{0}_{3\times3}\rangle\underline{\mathbf{Z}}_{21}^{\mathrm{T}} = \begin{bmatrix}\mathbf{I}_3 & \mathbf{BA}^{-1} \\ \mathbf{AB}^{-1} & \mathbf{I}_3\end{bmatrix}, \quad \mathbf{Z}_{12}\langle \mathbf{I}_3, \mathbf{0}_{3\times3}\rangle\underline{\mathbf{Z}}_{12}^{\mathrm{T}} = \begin{bmatrix}\mathbf{I}_3 & \mathbf{B}'\mathbf{A}'^{-1} \\ \mathbf{A}'\mathbf{B}'^{-1} & \mathbf{I}_3\end{bmatrix}, \tag{4.18a, b}$$

$$\mathbf{Z}_{12}\langle -\mathrm{i}\mathbf{I}_3, \mathrm{i}\mathbf{I}_3\rangle\underline{\mathbf{Z}}_{12}^{\mathrm{T}}\mathbf{II} = \langle -\mathrm{i}\overline{\mathbf{B}'\mathbf{A}'^{-1}} + (-\mathrm{i}\mathbf{BA}^{-1}), -\mathrm{i}\overline{\mathbf{A}'\mathbf{B}'^{-1}} + (-\mathrm{i}\mathbf{AB}^{-1})\rangle, \tag{4.19}$$

Eq. (3.3) implies that $\mathbf{A}^{\mathrm{T}}\{\overline{(-\mathrm{i}\mathbf{BA}^{-1})} - (-\mathrm{i}\mathbf{BA}^{-1})^{\mathrm{T}}\} = \mathbf{0}$, i.e., $-\mathrm{i}\,\mathbf{BA}^{-1}$ is Hermitian symmetric and so must be its inverse matrix $\mathrm{i}\,\mathbf{AB}^{-1}$. They are also known to be positive definite (Ting, 1996). Hence the right-hand side of Eq. (4.19), composed of one positive-definite diagonal block and another negative definite, is nonsingular, and so must also be $\mathbf{Z}_{12}$ and $\underline{\mathbf{Z}}_{12}$ on the left-hand side. Therefore $\mathbf{Z}_{12}^{-1}$ exists and Eq. (4.15) gives valid solutions of $\mathbf{T}'$ and $\overline{\mathbf{T}}$. One has

$$\mathbf{Z}'_\perp\mathbf{T}' = \mathbf{Z}'_\perp[\mathbf{I}_3, \mathbf{0}_{3\times3}]\mathbf{Z}_{12}^{-1}\mathbf{Z}_\perp = [\mathbf{Z}'_\perp, \mathbf{0}_{3\times3}]\langle \mathbf{I}_3, \mathbf{0}_{3\times3}\rangle\mathbf{Z}_{12}^{-1}\mathbf{Z}_\perp = \mathbf{Z}_{12}\langle \mathbf{I}_3, \mathbf{0}_{3\times3}\rangle\mathbf{Z}_{12}^{-1}\mathbf{Z}_\perp.$$

Alternative expressions of $\mathbf{T}$ and $\mathbf{T}'$ are found in the literature in terms of $\mathbf{B}$, $\mathbf{B}'$ and two real matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (Boem and Atluri, 1995; Ting, 1995)

$$\begin{aligned} & \mathbf{T} = -\mathbf{B}^{-1}(\mathbf{I}_3 - \mathrm{i}\boldsymbol{\beta})^{-1}(\boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta})\overline{\mathbf{B}}, \quad \mathbf{T}' = \mathbf{B}'^{-1}(\mathbf{I}_3 + \mathrm{i}\boldsymbol{\beta})^{-1}(\mathbf{I}_3 + \boldsymbol{\alpha})\mathbf{B}, \\ & \boldsymbol{\alpha} \equiv (\mathbf{L}' - \mathbf{L})(\mathbf{L}' + \mathbf{L})^{-1}, \quad \boldsymbol{\beta} \equiv (\mathbf{L}'^{-1} + \mathbf{L}^{-1})^{-1}(\mathbf{S}'\mathbf{L}'^{-1} - \mathbf{S}\mathbf{L}^{-1}), \end{aligned} \tag{4.20}$$

where $\mathbf{L}'$ and $\mathbf{S}'$ are defined in terms of $\mathbf{Z}'$ in the same way that $\mathbf{L}$ and $\mathbf{S}$ have been defined in terms of $\mathbf{Z}$. However, the expressions of Green's function in the two domains explicitly involve $\mathbf{Z}$ $\mathbf{Z}'$ and $\mathbf{Z}^{-1}$. Since these required matrices directly yield $\mathbf{T}$ and $\mathbf{T}'$ via Eq. (4.15) and (4.16), no particular advantage is gained by using Eq. (4.20) or similar alternative (real) forms for calculating $\mathbf{T}$ and $\mathbf{T}'$.

It is easy to see that the same pair of matrices $\mathbf{T}'$ and $\overline{\mathbf{T}}$ ensures the satisfaction of interface continuity conditions for the cases (i)–(iii) as well, since the algebraic structure of Eqs. (4.7a–c), (A.1b), (A.2b) and (A.3b) show patterns similar to Eqs. (4.7d) and (A.4b), but with fewer number of terms. The field equations of elasticity are always satisfied since the column vectors of $\mathbf{G}$ and $\mathbf{G}'$ are combinations of eigensolutions of the respective materials. It remains only to confirm that the solutions have the required singularity $2\pi\mathbf{I}_6$ at (b,h).

It is well known that, for an ND material, Eq. (4.8) has the discontinuity $2\pi\mathbf{I}_6$ at (b,h). When the type of material changes to D and ED, off-diagonal elements occur in the kernel matrix but these off-diagonal elements are bounded, continuous, periodic functions of θ. Therefore, material degeneracy does not change the discontinuity of the leading term $\mathbf{G}_\infty$ in Green's function, but only modifies its angular dependence. Furthermore, other terms in $\mathbf{G}$, and all terms in $\mathbf{G}'$, are nonsingular. Their apparent logarithmic and pole singularities are false singularities since they lie outside the region of validity of the respective functions.

Notice that the four cases (i)–(iv) depend only on the type of material of the region in which the singularity resides. Material degeneracy of the other region causes no *explicit* changes in $\mathbf{G}'$. That is, Eqs. (A1b)–(A4b) are valid regardless of the type of material of the lower region. However, material degeneracy still produces *implicit* changes in $\mathbf{G}'$ due to the higher-order eigenvectors in $\mathbf{Z}'$ and the off-diagonal elements in the kernel matrices.

Eq. (4.11), with $\mathbf{T}$ given by (4.4), presents Green's function of a half space for all types of material in a single concise expression. For a bimaterial, $\mathbf{T}$ is redefined by Eq. (4.15), along with $\mathbf{T}'$ for the lower region. A single concise expression may also be given for $\mathbf{G}'$ by using the double bracket symbol introduced in connection with Eq. (4.11):

$$\mathbf{G}' = \mathbf{Z}'[\![\| -\mathrm{i}\log[z'-\rho]\|\langle \mathbf{T}', \overline{\mathbf{T}}'\rangle]\!]\mathbf{Z}^{-1}. \tag{4.21}$$

More explicit expressions of $\mathbf{G}$ and $\mathbf{G}'$ are given in Appendix A.

5. The effect of material degeneracy on the singular stress field

Green's function has only one singularity, i.e., at the position of a concentrated line force and dislocation. The singularity results in the discontinuity $\mathbf{G}(r,\pi) - \mathbf{G}(r,-\pi) = 2\pi\mathbf{I}_6$ when circling a closed path around the singular point. For ND materials, the singular part of $\mathbf{G}$ is a combination of logarithmic functions, whereas $\mathbf{G}$ has pole singularities in the D and ED cases. These features are valid regardless of the domain shape and boundary or interface conditions, provided that the singularity lies not on the boundary or interface but in the interior region.

Let $\mathbf{G}$ be Green's function of a certain domain with boundary and interface conditions, and has the discontinuity $2\pi\mathbf{I}_6$ at $z=\rho$. Let $\mathbf{G}_\infty$ be defined by Eq. (4.8). Then $\mathbf{G}-\mathbf{G}_\infty$ is analytic in the domain and has no singularity. The prescribed boundary conditions of $\mathbf{G}$ and the boundary value of $\mathbf{G}_\infty$ determine that of $\mathbf{G}-\mathbf{G}_\infty$. In the interior of the domain, $\mathbf{G}-\mathbf{G}_\infty$ may be calculated accurately using numerical schemes because the function is regular. Hence accurate determination of Green's function for an arbitrary domain may be achieved by combining a numerical solution of $\mathbf{G}-\mathbf{G}_\infty$ with the exact expression of $\mathbf{G}_\infty$, Eq. (4.8).

Thus the effect of material degeneracy on the singular stress field of Green's function of general domains or bimaterials is identical to its effect on $\mathbf{G}_\infty$. Setting $\rho = 0$ in Eq. (4.8) and using Eq. (3.8), one obtains

$$\mathbf{G}_\infty = \log[r]\boldsymbol{\Gamma}\mathbf{II} + \boldsymbol{\Gamma}\mathbf{IIZ}\|\log[\cos\theta + \mu\sin\theta]\|\mathbf{Z}^{-1}. \tag{5.1}$$

While Green's function transforms χ_0 to the solution vector χ, the derivatives $\mathbf{Q}_6\mathbf{G}_{\infty,r}$ and $(1/r)\,\mathbf{Q}_6\mathbf{G}_{\infty,\theta}$ map χ_0 to $\{-\tau_{r_\theta}, -\sigma_\theta, -\tau_{\theta z}, \varepsilon_r, u_{\theta,r}, \gamma_{rz}\}^{\mathrm{T}}$ and $\{(u_{r,\theta} - u_\theta)/r, \varepsilon_\theta, \gamma_{\theta z}, \sigma_r, \tau_{r\theta}, \tau_{rz}\}^{\mathrm{T}}$, respectively. Using Eqs. (2.20) and (2.21) in Yin (2001), one obtains

$$\{-\tau_{r\theta}, -\sigma_\theta, -\tau_{\theta_z}, \varepsilon_r, u_{\theta,r}, \gamma_{rz}\}^{\mathrm{T}} = (1/r)\mathbf{Q}_6\boldsymbol{\Gamma}\mathbf{II}\chi_0, \tag{5.2}$$

$$\{(u_{r,\theta} - u_\theta)/r, \varepsilon_\theta, \gamma_{\theta z}, \sigma_r, \tau_r\theta, \tau_{rz}\}\}^{\mathrm{T}} = (1/r)\mathbf{Q}_6\boldsymbol{\Gamma}\mathbf{II}\mathbf{Z}(\mu\cos\theta - \sin\theta)/(\cos\theta + \mu\sin\theta)\mathbf{Z}^{-1}\chi_0, \tag{5.3}$$

where

$$\mathbf{Q}_6 \equiv \langle \mathbf{Q}_3, \mathbf{Q}_3 \rangle, \quad \mathbf{Q}_3 \equiv \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.4}$$

The kernel matrix in (5.3) has the diagonal elements μ_1^*, μ_2^*, μ_3^*, $\bar{\mu}_1^*$, $\bar{\mu}_2^*$ and $\bar{\mu}_3^*$, where

$$\mu_k^* \equiv (\mu_k\cos\theta - \sin\theta)/(\cos\theta + \mu_k\sin\theta) \quad (k = 1, 2, 3). \tag{5.5}$$

They are the eigenvalues of the same material referred to the local coordinate axes $\{r, \theta, z\}$. In the D and ED cases, the kernel matrix contains the following off-diagonal elements:

$$\mathrm{d}\mu^*/\mathrm{d}\mu = 1/(\cos\theta + \mu_0\sin\theta)^2, \quad \mathrm{d}^2\mu^*/\mathrm{d}\mu^2 = 2\sin\theta/(\cos\theta + \mu_0\sin\theta)^3. \tag{5.6a, b}$$

These additional elements show the effect of material degeneracy on the singular stress field of Green's function. Eq. (5.2) shows that the following stress components and displacement gradients of Green's function have a simple dependence on θ:

$$\{\tau_{r\theta}, \sigma_{\theta z}, \tau_{\theta z}\}^{\mathrm{T}} = -(1/r)\mathbf{Q}_3[\mathbf{S}^{\mathrm{T}}, -\mathbf{L}]\chi_0, \quad \{\varepsilon_r, u_{\theta,r}, \gamma_{rz}\}^{\mathrm{T}} = (1/r)\mathbf{Q}_3[\mathbf{H}, \mathbf{S}]\chi_0,$$

i.e., $r\tau_{\theta z}$ and $r\gamma_{rz}$ are constant in the region, while $r\{\tau_{r\theta}, \sigma_\theta\}^{\mathrm{T}}$ and $\{r\varepsilon_r, u_{\theta,r}\}^{\mathrm{T}}$ are obtained by the in-plane rotations of two constant vectors through the angle θ.

6. Higher-order singularities; singularities on the bimaterial interface or at the vertex of a multimaterial wedge

Some additional results on Green's functions of bimaterials may be derived from the solutions of the preceding section.

Green's function associated with double forces, line moments and higher-order singularities in displacements are obtained by differentiating the preceding Green's functions with respect to the position of the singularity. Differentiating Eqs. (4.11) and (4.21) with respect to b, one obtains the following for the upper and lower regions respectively

$$\partial\mathbf{G}/\partial b = \mathbf{Z}\|\mathrm{i}(z - \rho)^{-1}\|\mathbf{Z}^{-1} + \mathbf{Z}[\![\|\mathrm{i}(z - \bar{\rho})^{-1}\|\langle\mathbf{T}, \overline{\mathbf{T}}\rangle]\!]\mathbf{II}\mathbf{Z}^{-1}, \tag{6.1a}$$

$$\partial\mathbf{G}'/\partial b = \mathbf{Z}'[\![\|\mathrm{i}(z' - \rho)^{-1}\|\langle\mathbf{T}', \overline{\mathbf{T}}'\rangle]\!]\mathbf{Z}^{-1}, \tag{6.1b}$$

Differentiation with respect to h yields

$$\partial\mathbf{G}/\partial h = \mathbf{Z}\|\mathrm{i}\mu(z - \rho)^{-1}\|\mathbf{Z}^{-1} + \mathbf{Z}[\![\|\mathrm{i}\mu(z - \bar{\rho})^{-1}\|\langle\mathbf{T}, \overline{\mathbf{T}}\rangle]\!]\mathbf{II}\mathbf{Z}^{-1}, \tag{6.2a}$$

$$\partial\mathbf{G}'/\partial h = \mathbf{Z}'[\![\|\mathrm{i}\mu(z' - \rho)^{-1}\|\langle\mathbf{T}'\overline{\mathbf{T}}'\rangle]\!]\mathbf{Z}^{-1} \tag{6.2b}$$

Consider next the case when the singularity is moved from the upper half space to a position on the interface, which will be taken as the origin of the x–y plane. Then $b = h = 0$, so that $\rho_j = 0$ $(j = 1, 2, 3)$. Eqs. (4.11) and (4.21) reduce to

$$\mathbf{G} = \mathbf{Z}\| - \mathrm{i}\log[z]\|\mathbf{Z}^{-1} + \mathbf{Z}\| - \mathrm{i}\log[z]\|\langle\mathbf{T}, \overline{\mathbf{T}}\rangle\mathbf{IIZ}^{-1}. \tag{6.3a}$$

$$\mathbf{G}' = \mathbf{Z}'\|\mathrm{i}\log[z']\|\langle\mathbf{T}', \overline{\mathbf{T}}'\rangle\mathbf{Z}^{-1}. \tag{6.3b}$$

When the singularity is located on the interface, the bimaterial problem is algebraically symmetric with respect to the two half spaces. This symmetry is not transparent in Eqs. (6.3a,b). To obtain alternative expressions that manifest the symmetry, consider Green's function of the bimaterial with the singularity located at a point $(0, -h)$ in the lower half space, and then take the limit $h \to 0$. One obtains the following expressions instead of (6.3a,b):

$$\mathbf{G}' = \mathbf{Z}'\| - \mathrm{i}\log[z']\|\mathbf{Z}'^{-1} + \mathbf{Z}'\| - \mathrm{i}\log[z']\|\langle\mathbf{R}', \overline{\mathbf{R}}'\rangle\mathbf{IIZ}'^{-1}. \tag{6.4a}$$

$$\mathbf{G} = \mathbf{Z}\| - \mathrm{i}\log[z]\|\langle\mathbf{R}, \overline{\mathbf{R}}\rangle\mathbf{Z}'^{-1}, \tag{6.4b}$$

where $\mathbf{R}$ and $\overline{\mathbf{R}}'$ are solutions of the equation $\overline{\mathbf{Z}}'_{\perp}\overline{\mathbf{R}}' + \mathbf{Z}_{\perp}\mathbf{R} = \mathbf{Z}'_{\perp}$, or,

$$\mathbf{Z}_{12}\begin{Bmatrix}\mathbf{R}' \\ \overline{\mathbf{R}}\end{Bmatrix} = \overline{\mathbf{Z}}'_{\perp}. \tag{6.5}$$

One may combine Eqs. (6.5) and (4.14) into either one of the following two equivalent relations:

$$\begin{bmatrix}\mathbf{T}' & \mathbf{R}' \\ \overline{\mathbf{T}} & \overline{\mathbf{R}}\end{bmatrix} = \mathbf{Z}_{12}^{-1}\mathbf{Z}_{21} = \begin{bmatrix}\mathbf{R} & \mathbf{T} \\ \overline{\mathbf{R}}' & \overline{\mathbf{T}}'\end{bmatrix}^{-1} \tag{6.6a, b}$$

Then

$$\langle\mathbf{R}, \overline{\mathbf{R}}\rangle\mathbf{Z}'^{-1} = \langle[\mathbf{I}_3, \mathbf{0}_{3\times3}]\mathbf{Z}_{21}^{-1}\mathbf{Z}'_{\perp}, [\mathbf{0}_{3\times3}, \mathbf{I}_3]\mathbf{Z}_{12}^{-1}\overline{\mathbf{Z}}'_{\perp}\rangle\mathbf{Z}'^{-1} = \begin{bmatrix}[\mathbf{I}, \mathbf{0}]\mathbf{Z}_{21}^{-1} & [\mathbf{Z}'_{\perp}, \overline{\mathbf{Z}}'_{\perp}] \\ [\mathbf{0}, \mathbf{I}]\mathbf{Z}_{12}^{-1} & [\mathbf{Z}'_{\perp}, \overline{\mathbf{Z}}'_{\perp}]\end{bmatrix}\mathbf{Z}'^{-1} = \begin{bmatrix}[\mathbf{I}, \mathbf{0}]\mathbf{Z}_{21}^{-1} \\ [\mathbf{I}, \mathbf{0}]\mathbf{Z}_{12}^{-1}\end{bmatrix}, \tag{6.7a}$$

$$\mathbf{Z}^{-1} + \langle\mathbf{T}, \overline{\mathbf{T}}\rangle\mathbf{IIZ}^{-1}\begin{bmatrix}\mathbf{I}_3 & \mathbf{T} \\ \overline{\mathbf{T}} & \mathbf{I}_3\end{bmatrix}\mathbf{Z}^{-1} = \begin{bmatrix}[\mathbf{I}, \mathbf{0}]\mathbf{Z}_{21}^{-1} & [\mathbf{Z}_{\perp}, \overline{\mathbf{Z}}_{\perp}] \\ [\mathbf{0}, \mathbf{I}]\mathbf{Z}_{12}^{-1} & [\mathbf{Z}_{\perp}, \overline{\mathbf{Z}}_{\perp}]\end{bmatrix}\mathbf{Z}^{-1} = \begin{bmatrix}[\mathbf{I}, \mathbf{0}]\mathbf{Z}_{21}^{-1} \\ [\mathbf{0}, \mathbf{I}]\mathbf{Z}_{12}^{-1}\end{bmatrix}. \tag{6.7b}$$

The last expressions of (6.7a,b) are identical. Hence Eq. (6.4b) is identical to (6.3a) and so is (6.4a) to (6.3b). Thus, Green's function of bimaterials with an interfacial singularity may be expressed in a symmetric manner by Eqs. (6.3b) and (6.4b), respectively, in the lower and upper half spaces, i.e.,

$$\mathbf{G}' = \mathbf{Z}'\| - \mathrm{i}\log[z']\|\mathbf{P}_1, \quad \mathbf{G} = \mathbf{Z}\| - \mathrm{i}\log[z]\|\mathbf{P}_2, \tag{6.8a, b}$$

where

$$\mathbf{P}_1 \equiv \langle\mathbf{T}', \overline{\mathbf{T}}'\rangle\mathbf{Z}^{-1} = \begin{bmatrix}[\mathbf{I}, \mathbf{0}]\mathbf{Z}_{12}^{-1} \\ [\mathbf{0}, \mathbf{I}]\mathbf{Z}_{21}^{-1}\end{bmatrix} \quad \mathbf{P}_2 \equiv \langle\mathbf{R}, \overline{\mathbf{R}}\rangle\mathbf{Z}'^{-1} = \begin{bmatrix}[\mathbf{I}, \mathbf{0}]\mathbf{Z}_{21}^{-1} \\ [\mathbf{0}, \mathbf{I}]\mathbf{Z}_{12}^{-1}\end{bmatrix}. \tag{6.9a, b}$$

Using Eq. (3.7), one may establish the following relation between $\mathbf{P}_1$ and $\mathbf{P}_2$:

$$\boldsymbol{\Gamma}'\mathbf{IIZ}'\mathbf{P}_1 = \mathbf{Z}'\langle-\mathrm{i}\mathbf{I}_3, \mathrm{i}\mathbf{I}_3\rangle\mathbf{P}_1 = \mathbf{Z}\langle-\mathrm{i}\mathbf{I}_3, \mathrm{i}\mathbf{I}_3\rangle\mathbf{P}_2 = \boldsymbol{\Gamma}\mathbf{IIZP}_2. \tag{6.10}$$

Eqs. (6.8a,b) yield $\mathbf{G}$ and $\mathbf{G}'$ on the two segments of the interface separated by the singularity

$$(\mathbf{G}|_{\theta=0}) = \log[r]\boldsymbol{\Gamma}\mathbf{\Pi}\mathbf{Z}\mathbf{P}_2 = \mathbf{G}'|_{\theta=0} = \log[r]\boldsymbol{\Gamma}'\mathbf{\Pi}\mathbf{Z}'\mathbf{P}_1,$$

$$(\mathbf{G}|_{\theta=\pi}) = \log[r]\boldsymbol{\Gamma}\mathbf{\Pi}\mathbf{Z}\mathbf{P}_2 + \pi\mathbf{Z}\mathbf{P}_2, \quad (\mathbf{G}'|_{\theta=-\pi}) = \log[r]\boldsymbol{\Gamma}'\mathbf{\Pi}\mathbf{Z}'\mathbf{P}_1 - \pi\mathbf{Z}'\mathbf{P}_1.$$

Hence $\mathbf{G}$ and $\mathbf{G}'$ have the required discontinuity across the negative x-axis

$$(\mathbf{G}|_{\theta=\pi}) - (\mathbf{G}|_{\theta=\pi}) = \pi(\mathbf{Z}\mathbf{P}_2 + \mathbf{Z}'\mathbf{P}_1) = 2\pi\mathbf{I}_6.$$

We rewrite Eqs. (6.8a,b) using the polar coordinates

$$\mathbf{G}' = \log[r]\boldsymbol{\Gamma}'\mathbf{\Pi}\mathbf{Z}'\mathbf{P}_1 + \boldsymbol{\Gamma}'\mathbf{\Pi}\mathbf{Z}'\|\log[\cos\theta + \mu'\sin\theta]\|\mathbf{P}_1, \tag{6.11a}$$

$$\mathbf{G} = \log[r]\boldsymbol{\Gamma}\mathbf{\Pi}\mathbf{Z}\mathbf{P}_2 + \boldsymbol{\Gamma}\mathbf{\Pi}\mathbf{Z}\|\log[\cos\theta + \mu\sin\theta]\|\mathbf{P}_2, \tag{6.11b}$$

The two expressions share a common first term according to Eq. (6.10), while their second terms have constant values on radial lines. Across each radial line, the components of the traction vector, as well as $\varepsilon_r, u_{\theta,r}$ and γ_{rz}, are given by the tangential derivative of $\boldsymbol{\chi}$, to which the second terms in the last two expressions make no contribution. One has

$$r\{-\tau_{r\theta}, -\sigma_\theta, -\tau_{\theta_z}, \epsilon_r, u_{\theta,r}, \gamma_{rz}\}^{\mathrm{T}} = \mathbf{Q}_6\boldsymbol{\Gamma}\mathbf{\Pi}\mathbf{Z}\mathbf{P}_2\boldsymbol{\chi}_0 = \mathbf{Q}_6\boldsymbol{\Gamma}'\mathbf{\Pi}\mathbf{Z}'\mathbf{P}_1\boldsymbol{\chi}_0, \tag{6.12a}$$

$$\begin{aligned} &r\{\sigma_r, \tau_{r\theta}, \tau_{rz}, (u_{r,\theta} - u_\theta)/r, \varepsilon_\theta, \gamma_{\theta_z}\}^{\mathrm{T}} \\ &\quad = \mathbf{Q}_6\boldsymbol{\Gamma}\mathbf{\Pi}\mathbf{Z}\|(\mu\cos\theta - \sin\theta)/(\mu\sin\theta + \cos\theta)\|\mathbf{P}_2\boldsymbol{\chi}_0 \quad \text{(upper region)}, \end{aligned} \tag{6.12b}$$

$$\begin{aligned} &r\{\sigma_r, \tau_{r\theta}, \tau_{rz}, (u_{r,\theta} - u_\theta)/r, \varepsilon_\theta, \gamma_{\theta_z}\}^{\mathrm{T}} \\ &\quad = \mathbf{Q}_6'\mathbf{\Pi}\mathbf{Z}'\|(\mu'\cos\theta - \sin\theta)/(\mu'\sin\theta + \cos\theta)\|\mathbf{P}_1\boldsymbol{\chi}_0 \quad \text{(lower region)} \end{aligned} \tag{6.12c}$$

6.1. Bimaterial with a slanted interface

Now consider another biomaterial, made of the same pair of materials having the same orientations as the previous one, but with the interface slanted at an angle α. It is rather surprising that Green's function for an interface singularity is essentially identical for these two bimaterials. To show this, consider the two segments of the interface $y = x\tan\alpha$ separated by the singularity at the origin. On the segment $\theta = \alpha$, Eqs. (6.11a), (6.11b) and (6.10) yield a constant difference of $\mathbf{G}$ and $\mathbf{G}'$:

$$(\mathbf{G} - \mathbf{G}')|_{\theta=\alpha} = \mathbf{C} \equiv \boldsymbol{\Gamma}\mathbf{\Pi}\mathbf{Z}\|\log[\cos\alpha + \mu\sin\alpha]\|\mathbf{P}_2 - \boldsymbol{\Gamma}'\mathbf{\Pi}\mathbf{Z}'\|\log[\cos\alpha + \mu\sin\alpha]\|\mathbf{P}_1.$$

Similarly, $\mathbf{G}|_{\theta=\alpha+\pi}$ and $\mathbf{G}'|_{\theta=\alpha-\pi}$ differ by $\mathbf{C} + 2\pi\mathbf{I}_6$. The additive constant matrix $\mathbf{C}$ has no effect on the stress and strain fields. By adding an appropriate $\mathbf{C}$ to the right-hand side of Eqs. (6.8a,b) and (6.11b), while leaving (6.8a,b) and (6.11a) unchanged, one may enforce displacement continuity across $\theta = \alpha$. This gives Green's function of the second bimaterial, i.e., one with the interface slanted at an arbitrary angle. Of course the interfacial tractions are different for the two bimaterials, since they are the evaluations of the same functions on different radial lines. In the two sectors $\alpha < \theta < \pi$ and $-\pi + \alpha < \theta < 0$, the original and the slanted bimaterials share common materials, and indeed they have the same stress and strain fields in these sectors.

6.2. Multimaterial wedges with a singularity at the vertex

More generally, consider a multi-material wedge composed of N distinct anisotropic sectors joined along $N - 1$ radial interface lines $\theta = \theta_1, \theta_2 \ldots, \theta_{N-1}$ and having two exterior boundary edges $\theta = 0$ and

$\theta = \theta^* \equiv \theta_N$, where the boundary conditions $(\mathbf{K}_0\boldsymbol{\chi}|_{\theta=0}) = 0$ and $(K_N\boldsymbol{\chi}|_{\theta=\theta^*}) = 0$, are imposed in a manner similar to Eq. (4.3) (Yin, 2003a). The material in the s th sector, which lies between the interfaces $\theta = \theta_{s-1}$ and a $\theta = \theta_s$, has the matrix of eigenvectors $\mathbf{Z}^{(s)}$ and a set of three eigenvalues $\{\mu_\perp^{(s)}\}$. In the successive sectors, we let

$$\mathbf{G}^{(1)} = \mathbf{Z}^{(1)}\| - \mathrm{i}\log[z]\|^{(1)}\mathbf{P}^{(1)}\mathbf{D}, \quad \mathbf{G}^{(p)} = \mathbf{Z}^{(p)}\| - \mathrm{i}\log[z]\|^{(p)}\mathbf{P}^{(p)}\mathbf{D} + \mathbf{C}^{(p)}\mathbf{D}, \tag{6.13a, b}$$

where $\mathbf{D}$ and $\mathbf{P}^{(p)}(p = 1, 2, \ldots, N)$ are 6×6 constant matrices to be determined

$$\begin{aligned}\mathbf{C}^{(p)} \equiv &-\sum_{1\leqslant r\leqslant p} \mathbf{Z}^{(r)}\| - \mathrm{i}\log[\cos\theta_{r-1} + \mu\sin\theta_{r-1}]\|^{(r)}\mathbf{P}^{(r)} \\ &+ \sum_{1\leqslant r\leqslant p1} \mathbf{Z}^{(r)}\| - \mathrm{i}\log[\cos\theta_r + \mu\sin\theta_r]\|^{(r)}\mathbf{P}^{(r)} \quad (p = 2, \ldots, N)\end{aligned} \tag{6.13c}$$

and $\mathbf{Z}^{(r)} = [\mathbf{Z}_\perp^{(r)}, \overline{\mathbf{Z}}_\perp^{(r)}]$ is the base matrix of the rth sector. The kernel matrix $\| - \mathrm{i}\log[z]\|^{(r)}$ contains the eigenvalues of the rth sector as the parameters. The interface continuity conditions $\mathbf{G}^{(p)} = \mathbf{G}^{(p-1)}$ on $\theta = \theta_{p-1}$ yield equalities similar to (6.10)

$$\mathbf{Z}^{(p)}\langle -\mathrm{i}\mathbf{I}_3, \mathrm{i}\mathbf{I}_3\rangle\mathbf{P}^{(p)} = \mathbf{Z}^{(p-1)}\langle -\mathrm{i}\mathbf{I}_3, \mathrm{i}\mathbf{I}_3\rangle\mathbf{P}^{(p-1)} \quad (p = 2, \ldots, N), \tag{6.14}$$

which may be satisfied by choosing

$$\mathbf{P}^{(p)} \equiv \begin{bmatrix} [\mathbf{I}_3, \mathbf{0}][\mathbf{Z}_\perp^{(p)}, \overline{\mathbf{Z}}_\perp^{(\mathbf{p+1})}]^{-1} \\ [\mathbf{0}, \mathbf{I}_3][\mathbf{Z}_\perp^{(\mathbf{p+1})}, \overline{\mathbf{Z}}_\perp^{(\mathbf{p})}]^{-1} \end{bmatrix} \equiv \begin{bmatrix} [\mathbf{I}_3, \mathbf{0}][\mathbf{Z}_\perp^{(p)}, \overline{\mathbf{Z}}_\perp^{(\mathbf{p-1})}]^{-1} \\ [\mathbf{0}, \mathbf{I}_3][\mathbf{Z}_\perp^{(\mathbf{p-1})}, \overline{\mathbf{Z}}_\perp^{(\mathbf{p})}]^{-1} \end{bmatrix}. \tag{6.15a, b}$$

Then

$$\begin{aligned}\Delta\mathbf{G} \equiv \mathbf{G}^{(N)}|_{\theta=\theta^*} = \mathbf{G}^{(1)}|_{\theta=0} &= \mathbf{Z}^{(N)}\| - \mathrm{i}\log[\cos\theta^* + \mu\sin\theta^*]\|^{(N)}\mathbf{P}^{(N)}\mathbf{D} + \mathbf{C}^{(N)}\mathbf{D} \\ &= \left\{\sum_{1\leqslant p\leqslant N} \mathbf{Z}^{(p)}\| - \mathrm{i}\log[\cos\theta_{p-1} + \mu\sin\theta_{p-1}]\|^{(p)}\right\}\mathbf{P}^{(p)} \\ &\quad + \left\{\sum_{1\leqslant p\leqslant N} \mathbf{Z}^{(p)}\| - \mathrm{i}\log[\cos\theta_p + \mu\sin\theta_p]\|^{(p)}\mathbf{P}^{(p)}\right\}\mathbf{D}.\end{aligned} \tag{6.16}$$

The multimaterial wedge is called *closed* if $\theta_N = 2\pi$ and if the Nth sector and the first sector are perfectly bonded along the coalescing radial lines $\theta = 0$ and $\theta = 2\pi$. Substituting Eq. (6.15) and $\Delta\mathbf{G} = 2\pi\mathbf{I}_6$ into Eq. (6.16), one obtains the constant matrix $\mathbf{D}$. Then $\mathbf{G}^{(p)}$ is given by Eqs. (6.13a–c), with $\mathbf{P}^{(p)}$ expressed by (6.15). If the closed wedge has only two half-plane sectors, then it is a bimaterial and Eqs. (6.13) and (6.15a,b) reduce to (6.8) and (6.9).

In the contrary case of an open wedge, the two boundary edges $\theta = 0$ and $\theta = \theta^* < 2\pi$ are subjected to the boundary conditions

$$\mathbf{K}_0\mathbf{Z}^{(1)}\mathbf{P}^{(1)}\mathbf{D} = \mathbf{0}, \tag{6.17a}$$

$$\begin{aligned}\mathbf{K}_N - &\left\{\sum_{2\leqslant p\leqslant N} \mathbf{Z}^{(p)}\| - \mathrm{i}\log[\cos\theta_{p-1} + \mu\sin\theta_{p-1}]\|^{(p)}\right\}\mathbf{P}^{(p)} \\ &+ \left\{\sum_{2\leqslant p\leqslant N} \mathbf{Z}^{(p)}\| - \mathrm{i}\log[\cos\theta_p + \mu\sin\theta_p]\|^{(p)}\mathbf{P}^{(p)}\right\}\mathbf{D} \\ &= -\mathbf{K}_N\Delta\mathbf{G} = -2\pi\mathbf{K}_N.\end{aligned} \tag{6.17b}$$

The solution of these equations gives the undetermined matrix $\mathbf{D}$ of Eq. (6.13a,b).

Notice that Eq. (6.8) for a bimaterial with an interfacial singularity, and Eq. (6.13) for a multimaterial wedge with a singularity at the vertex, when restricted to a single sector, have the structure of Green's function of an infinite space, Eq. (4.8). The stress and strain fields are given by Eqs. (6.12a–c), which possess the prominent feature that the traction vector and the tangential strains have identical distributions on all radial planes except for the rotation $\mathbf{Q}_3$. By enforcing Eq. (6.10) on the bimaterial and Eq. (6.14) on the multimaterial wedge, the cylindrical components of the traction vector and the radial derivatives of u_r, u_θ and w have the same simple θ-dependencies, Eq. (6.12a), in all sectors despite the differences in materials. All that is left for patching up the solutions of the various sectors is to enforce the interfacial continuity of the normal displacement, and this is achieved by including the constant vector $\mathbf{C}^{(p)}$ in Eq. (6.13b). From this follows, in particular, the indifference of the interfacial Green's function of a bimaterial wedge to the slanting of the interface.

7. Concluding remarks

Degenerate and ED materials do not possess three complex conjugate pairs of zeroth-order eigenvectors. The deficiency is made up by higher-order eigenvectors, obtainable analytically from the zeroth-order eigenvectors according to the derivative rule. The general solution of two-dimensional anisotropic elasticity has the expression $\boldsymbol{\chi} = \mathbf{Z}\|f(x+\mu y)\|\mathbf{c}$, where the upper 3×3 diagonal block of the kernel matrix $\|f(x+\mu y)\|$ is given by Eqs. (3.14a–d) for the various types of anisotropic materials, and the base matrix $\mathbf{Z}$ may contain higher-order eigenvectors.

These results are fundamental to the derivation of Green's functions of various domains involving D and ED materials. For Green's function of the infinite space, $\mathbf{G}_\infty$, material degeneracy affects Eq. (4.8) only implicitly through the appearance of the higher-order eigenvectors and of the off-diagonal elements in $\|-\mathrm{i}\log[x+\mu y]\|$. Notice that $\mathbf{G}_\infty$ of ND materials leads to those of D and ED materials straightforwardly according to the derivative rule. For bounded regions or multimaterials, degeneracy affects $\mathbf{G}$ not only implicitly, but also explicitly, resulting in additional terms in Green's function, as required by the boundary and interface conditions. For a half-space, and for bimaterials with a planar interface, the additional terms in Green's function of the D and ED cases form two groups, and the rule of association between the two groups is given by Eqs. (4.9) and (4.10).

The analytical complications of D and ED cases result from the simple fact that a kth-order eigensolution involves not only the eigenvector of the same order, but also all lower-order eigenvectors, as shown by Eqs. (2.15c) and (2.16c), respectively, for normal and abnormal materials. These relations are purely algebraic, and they determine the structure of Green's functions of D and ED materials. Although geometrical metaphor in terms of "image singularities" may offer supporting arguments for the results already obtained by algebraic analysis (often in the cases with strong symmetry, such as isotropic and orthotropic materials), geometrical reasoning alone can neither suggest nor foresee such results in more complicated D and ED cases. It does not lead the analysis as a cart does not draw a horse.

In the present formalism, the line force singularities and dislocation singularities are treated in a unified way. Setting $\boldsymbol{\chi}_0 = 2\mathrm{Re}[\boldsymbol{\xi}_k]$, where $\boldsymbol{\xi}_k$ is the kth eigenvector (whether of the zeroth or higher order), then Eq. (4.11) yields

$$\mathbf{G}\boldsymbol{\chi}_0 = \mathrm{Re}[\mathbf{Z}_\perp\ (k\text{th column of } \|-\mathrm{i}\log[z]\|_\perp)] + \mathrm{Re}[\overline{\mathbf{Z}}_\perp [\![\|-\mathrm{i}\log[z-\bar{p}]\|_\perp\ (k\text{th column of } \mathbf{T})]\!]].$$

The second term of this expression generally involves *all* eigenvectors. Hence, all eigensolutions are coupled in Green's function through the boundary and interface matrices $\mathbf{T}$ and $\mathbf{T}'$.

A prominent effect of material degeneracy is that it affects the angular variation of the singular stress field, as shown explicitly by Eq. (5.3). This effect is material-specific, independent of the domain shape and boundary and interface conditions. Yet it has been claimed that a D or ED material may always be approximated by a nondegenerate one with proximate but distinct eigenvalues through slight perturbation of the elastic constants, and it is further claimed that all 2-D anisotropic elasticity problems can be handled in the framework of Stroh's formalism of nondegenerate materials, with practically no need of the new forms of eigensolutions for the various degenerate cases. A response to this argument is given in the following.

All isotropic materials are degenerate, and their general solution is uncoupled into an anti-plane deformation and a plane deformation. The latter is given by Goursat's representation of the stress function $F = \mathrm{Re}[\zeta + \bar{z}\phi]$, where ζ and ϕ are analytic functions of $z = x + \mathrm{i}y$. Virtually the entire literature of plane isotropic elasticity depends on this form of the general solution. No writer has proposed to lift the degeneracy by perturbing the isotropic elastic constants so that the material eigenvalues may become unequal. No one has suggested that the works of Muskhelishvili (1953) and others are thereby rendered superfluous. In problems involving cracks and multi-material singularities, solutions of D and ED materials in general, and of isotropic materials in particular, possess complex and variegated analytical properties that may or may not remain intact under perturbations that affect the multiplicity of eigenvalues.

Even practical problems benefit from rigorous formulation. A simple exact analysis sometimes gives key parameters although not complete solutions. For all five types of anisotropic materials, symbolic algorithms have been developed in *Mathematica* to obtain exact expressions of the eigenvectors, the pseudo-metric $\boldsymbol{\Omega}$, the constant tensor $\boldsymbol{\Gamma}$ (containing the Stroh–Barnett–Lothe tensors as submatrices), and boundary and interface matrices ($\mathbf{T}$ and $\mathbf{T}'$ of Section 4). For problems including interface cracks and multi-material junctions, the singular part of the local stress field, if not the full elasticity solution, may be obtained rigorously and the results depend on material degeneracy. Such exact results cannot be obtained after the perturbation of elastic constants, which essentially replaces a system of coupled differential equations by an uncoupled system of lower-order equations. Besides the fundamental question as to whether the replacement may affect the behavior of solutions near the singularities, there is also the common sense question: why should the elastic constants be perturbed first before applying the numerical methods? The explicit expressions of D and ED materials, as given in this paper, allow the problem to be formulated in no less a proper manner in comparison with ND materials. Once formulated, there is no noticeable *computational* difference between numerical tasks involving degenerate materials and the perturbed problems involving ND materials. Thus the latter method fulfills no real need and offers no computational advantage, but suffers from questionable validity and lack of transparency of analysis.

The complex formalism greatly facilitates the analytical derivation in plane elasticity. The constant complex matrices including $\mathbf{Z}, \mathbf{Z}', \mathbf{Z}_{12}, \mathbf{Z}^{-1}, \boldsymbol{\Gamma}, \mathbf{T}, \mathbf{T}'$ and $\mathbf{P}^{(p)}$ are all easily calculated by using *Mathematica*, which manipulates complex entities just as easily as real entities. Analytical expressions of the kernel matrices and of Green's functions are also obtained explicitly by using the symbolic algebraic capabilities of *Mathematica*. From both theoretical and computational points of view, there is no compelling reason for favoring real expressions and real entities over complex ones in the intermediate steps of the analysis. Complex formalism cannot be dispensed with since the material eigenvalues, eigenvectors and the analytic functions in the kernel matrices are intrinsically complex. Then there is the truism that every complex entity canalways be replaced by two real entities if one so desires. Therefore, getting intermediate expressions in terms of certain real matrices brings only superficial modifications (e.g., $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of Eq. (4.21) instead of the bimaterial matrices $\mathbf{T}$ and $\mathbf{T}'$). Problems of plane *isotropic* elasticity have been formulated and solved using complex formalisms and techniques without apologies. The matter should not be different in anisotropic elasticity.

Recently, Yin (2003b,c) gave a complete analysis of the related subject on symmetric and unsymmetric anisotropic laminated plates regardless of laminate degeneracy, and determined the general solution for all eleven distinct types of laminates. Green's functions for the infinite plate, a semi-infinite plate with various edge conditions, and an infinite plate containing an elliptical hole have also been given for all cases of bending-extension coupling and laminate degeneracy (Yin, 2004b). Based on these works, the analysis of the present paper may be easily modified to obtain Green's function of a *bimaterial* laminate formed by joining two dissimilar, anisotropic, semi-infinite plates, and subjected to various types of concentrated forces and moments (both tangential and normal to the plate) as well as dislocations in displacements and slopes. For this Greeen's function, the formal expressions of the present equations (4.11) and (4.21) remain valid in the upper and lower half planes, respectively, and the interfacial matrices $\mathbf{T}$ and $\mathbf{T}'$ are still determined by Eq. (4.15). However, one must use the 8×8 base matrices $\mathbf{Z}$ and $\mathbf{Z}'$ for anisotropic laminates instead of the present 6×6 base matrices for 2-D anisotropic elasticity, and the symbols $\|\,\|$ and $[[\,]]$ in Eqs. (4.11) and (4.21) must be redefined for each one of the eleven distinct types of non-degenerate, degenerate, extra-degenerate and ultra-degenerate laminates (Yin, 2003b,c), for the same reason that these symbols have been defined presently for the five classes of ND, D and ED anisotropic elastic materials according to Eqs. (3.13a,b,c,d), (4.9) and (4.10).

Appendix A. Green's functions of anisotropic bimaterials

The bimaterials matrices $\mathbf{T}$ and $\mathbf{T}'$ are determined by Eq. (4.16) and by Eq. (4.21). μ_0 denotes a multiple eigenvalue.

Case (i)
Upper region material ND

$$\mathbf{Z}^{-1}\mathbf{G}\mathbf{Z} = \|-\mathrm{i}\log[x+\mu y-\mu h]\| + \sum_{1\leqslant j\leqslant 3}\|-\mathrm{i}\log[x+\mu y-\bar{\mu}_j h]\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\langle\varDelta_j,\varDelta_j\rangle\mathbf{II}, \tag{A.1a}$$

$$\mathbf{Z}^{-1}\mathbf{G}'\mathbf{Z} = \sum_{1\leqslant j\leqslant 3}\|-\mathrm{i}\log[x+\mu' y-\mu_j h]\|\langle\mathbf{T}',\overline{\mathbf{T}}'\rangle\langle\varDelta_j,\varDelta_j\rangle. \tag{A.1b}$$

Case (ii)
Upper region material D-normal, with $\{\mu\}_\perp = \{\mu_1,\mu_0,\mu_0\}$

$$\mathbf{Z}^{-1}\mathbf{G}\mathbf{Z} = \|-\mathrm{i}\log[x+\mu y-\mu h]\| + \sum_{1\leqslant j\leqslant 3}\|-\mathrm{i}\log[x+\mu y-\bar{\mu}_j h]\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\langle\varDelta_j,\varDelta_j\rangle\mathbf{II}$$
$$+\|\mathrm{i}h(x+\mu y-\bar{\mu}_0 h)^{-1}\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\langle\varDelta_{23},\varDelta_{23}\rangle\mathbf{II}, \tag{A.2a}$$

$$\mathbf{Z}'^{-1}\mathbf{G}'\mathbf{Z} = \sum_{1\leqslant j\leqslant 3}\|-\mathrm{i}\log[x+\mu' y-\mu_j h]\|\langle\mathbf{T}',\overline{\mathbf{T}}'\rangle\langle\varDelta_j,\varDelta_j\rangle + \|\mathrm{i}h(x+\mu' y-\mu_0 h)^{-1}\|\langle\mathbf{T}'\overline{\mathbf{T}}'\rangle\langle\varDelta_{23},\varDelta_{23}\rangle. \tag{A.2b}$$

Case (iii)
Upper region material D-abnormal (triple abnormal eigenvalue μ_0)

$$\mathbf{Z}^{-1}\mathbf{G}\mathbf{Z} = \|-\mathrm{i}\log[x+\mu y-\mu h]\| + \|-\mathrm{i}\log[x+\mu y-\bar{\mu}_0 h]\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\mathbf{II}$$
$$+\|\mathrm{i}h(x+\mu y-\bar{\mu}_0 h)^{-1}\|\langle\mathbf{T},\overline{\mathbf{T}}\rangle\langle 2\varDelta_{23},2\varDelta_{23}\rangle\mathbf{II}; \tag{A.3a}$$

$$\mathbf{Z}'^{-1}\mathbf{G}'\mathbf{Z} = \|-\mathrm{i}\log[x+\mu' y-\mu_0 h]\|\langle\mathbf{T}'\overline{\mathbf{T}}'\rangle + \|\mathrm{i}h(x+\mu' y-\mu_0 h)^{-1}\|\langle\mathbf{T}'\overline{\mathbf{T}}'\rangle\langle 2\varDelta_{23},2\varDelta_{23}\rangle. \tag{A.3b}$$

Case (iv)
Upper region material extra-degenerate (triple normal eigenvalue μ_0)

$$\mathbf{Z}^{-1}\mathbf{G}\mathbf{Z} = \| - \mathrm{i}\log[x + \mu y - \mu h]\| + \| - \mathrm{i}\log[x + \mu y - \bar{\mu}_0 h]\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle \mathbf{II}$$
$$+ \|\mathrm{i}h(x + \mu y - \mu_0 h)^{-1}\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle\langle \varDelta_{12} + 2\varDelta_{23}, \varDelta_{12} + 2\varDelta_{23}\rangle \mathbf{II};$$
$$+ \|\mathrm{i}h^2(x + \mu y - \bar{\mu}_0 h)^{-2}\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle\langle \varDelta_{13}, \varDelta_{13}\rangle \mathbf{II}, \tag{A.4a}$$

$$\mathbf{Z}'^{-1}\mathbf{G}'\mathbf{Z} = \| - \mathrm{i}\log[x + \mu' y - \mu_0 h]\|\langle \mathbf{T}', \overline{\mathbf{T}}'\rangle + \|\mathrm{i}h(x + \mu' y - \mu_0 h)^{-1}\|\langle \mathbf{T}', \overline{\mathbf{T}}'\rangle\langle \varDelta_{12} + 2\varDelta_{23}, \varDelta_{12} + 2\varDelta_{23}\rangle$$
$$+ \|\mathrm{i}h^2(x + \mu' y - \mu_0 h)^{-2}\|\langle \mathbf{T}, \overline{\mathbf{T}}\rangle\langle \varDelta_{13}, \varDelta_{13}\rangle. \tag{A.4b}$$

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